

Single thermal zone balance solved by Transfer Function Method

Giorgio Beccali, Maurizio Cellura, Valerio Lo Brano*, Aldo Orioli

DREAM, Dipartimento di Ricerche Energetiche ed Ambientali, Università di Palermo, Viale delle Scienze, 90128 Palermo, Italy

Received 25 November 2004; received in revised form 9 February 2005; accepted 13 February 2005

Abstract

We present an algorithm that uses the Z-transform operator to face the problem of heat transmission in a single thermal zone composed by multilayered walls. The method is very flexible and could be adopted to calculate the transfer function coefficients able to simulate the thermal behaviour of a room in free floating. Knowing the transfer function coefficients, it is possible to simulate the dynamic profile of each inner surfaces temperature and furthermore of the inner air temperature.

The proposed algorithm is fully described granting maximum clarity. The explicitness of all steps of the calculus make possible the definition of a method that is able to vary all of the calculus parameters such as sampling g period, number of roots, number of poles or number of coefficients.

To assess the reliability of the algorithm, we carried out a comparison between simulation data obtained from our method, from Fourier steady-state algorithm and those obtained from TRNSYS.

© 2005 Elsevier B.V. All rights reserved.

Keywords: Transfer Function Method; Energy simulation in buildings; Poles; Roots; Sampling period

1. Introduction

The “ASHRAE” procedure called Transfer Function Method (TFM), largely diffused in designing and simulating HVAC, analyses the heat transmission problems using Z-transform. Z-transform is a suitable mathematical operator when simulation models have to deal with data inputs or outputs discrete in the time domain such as climatic data [1,2].

As well as the other mathematical procedures related to the study of dynamic thermal behaviour of buildings, TFM is an approximated mathematical approach because the exact solution would require an infinite number of calculus [3].

Excluding approximations linked to physical assumptions, the most important source of inaccuracy is due to the truncation of the infinite coefficients that constitute the Transfer Functions (TFs) [4,5]. When TFs are in the form of $\frac{\text{num}(z)}{\text{den}(z)}$, the absolute value of the coefficients decreases very quickly increasing the order of the addendum.

This particular feature, and the fact that in the employed equations, the coefficients are in an explicit form, make easier the truncation procedure of TFs.

To execute a correct truncation procedure, it is necessary to evaluate the effect in the numerical response linked to the insertion or the elimination of a coefficient. This evaluation can be obtained only having a large number of coefficients.

In the ASHRAE manuals, available data permit to carry out this evaluation only for the TFs of walls. On the contrary, TFs of cooling load are described by expressions like:

$$K(z) = \frac{v_0 + v_1 z^{-1}}{1 + w_1 z^{-1}} \quad (1)$$

which employs no more than three coefficients globally.

In fact, while in the calculus of the TFs for walls was used a rigorous mathematical approach, in the case of cooling load of a thermal zone was adopted a numerical procedure able to obtain not the TFs but only the time response of the system solicited by an unitary impulse signal. This response in the time domain, describing the nature of the system, is characterised by a slower time decay in presence of higher values of thermal inertia. Practically, to employ this function, we should select a large number of coefficients.

* Corresponding author. Tel.: +39 091 236 118; fax: +39 091 48 4425.

E-mail address: lo_brano@dream.unipa.it (V. Lo Brano).

URL: <http://www.dream.unipa.it>

Furthermore, in this case, the absolute value of the coefficients do not decrease quickly increasing their order.

When TFs are in the form of equation (1), the denominator contains terms linked exclusively to the system. In other words, $D(z)$ “contains” the system, to limit the number of coefficients make more approximate the description of the system. Not having the function $D(z)$, in the “ASHRAE” method, they obtained a denominator by using a procedure founded on the hypothesis that the response of the system contains an infinite series of exponential terms with a negative time constant. [6,7] Then, the procedure should be able to identify first dominant poles of TFs and making possible to write the denominator in the form:

$$D(z) = (1 - CR_1 z^{-1})(1 - CR_2 z^{-1})(1 - CR_3 z^{-1}) \quad (2)$$

In case of thermal zone with low value of thermal inertia, time decay of the response is so fast to prevent the calculus of CR_2 and CR_3 . This means that thermal zones delimited by walls with low thermal inertia, described by TFs with at least three or four poles, have a thermal behaviour that can be described employing an unique pole relate to CR_1 .

Clearly, in the above procedure, there is not a fully strict approach from the mathematical point of view. It depends from the employed method to solve the system's equations.

To solve these problems, authors present a method to solve in a strict mathematical approach a system of linear differential equations representing, under some physical hypothesis, the thermal dynamic behaviour of a thermal zone. This procedure determines the TFs of the system by using Z-transform, finding all the coefficients that we require, independently from their order, thermal inertia of the room and walls.

2. Thermal balance of the thermal zone

On the external surface, the thermal balance is simplified by using the concept of air–sol temperature. The inner surface presents a heat flux due to the convective exchange with the internal air and the radiative exchange with other surfaces. It also receives a thermal flux from the internal environment due to electric devices or cooling plants. The thermal exchange between the glass and the internal air is summarized by an exchange coefficient. Solar radiation is distributed by means of a simple linear model related to the extension of the inner surfaces.

In order to evaluate the accuracy of calculations carried out by using a given set of ZT coefficients, we have to compare them with a reference response coming from a procedure having a different mathematical background and even able to give the time continuous response of the system.

3. Solving the equation's system

If a system is solicited by an input signal $i(t)$ time variable, that produces an output signal $o(t)$, in the Z-transform domain, the link we have to determine is $G(z) = \frac{O(z)}{I(z)}$, in which $I(z)$ and $O(z)$ are the Z-transforms (ZT) of $i(t)$ and $o(t)$.

The calculus can be done using:

$I(s) = \text{LT}[i(t)]$, $O(s) = \text{LT}[o(t)]$ and the transfer function of the system in the Laplace domain $G(s)$.

In this case, the transfer function in the Z domain is:

$$G(z) = \frac{O(z)}{I(z)} = \frac{\text{ZT}[O(s)]}{\text{ZT}[I(s)]} = \frac{\text{ZT}[I(s)G(s)]}{\text{ZT}[I(s)]} = \frac{\text{num}(z)}{\text{den}(z)} \quad (3)$$

where $\text{num}(z)$ is a polynomial called *numerator* and $\text{den}(z)$ is a polynomial called *denominator*. Assuming a linear ramp as input signal, applying the Heaviside theorem, we obtain:

$$O(s) = \frac{1}{s^2} G(s) = \frac{1}{s^2} \frac{\text{NUM}(s)}{\text{DEN}(s)} = \frac{C_0}{s^2} + \frac{C_1}{s} + \sum \frac{\text{res}_n}{(s - s_n)} \quad (4)$$

where s_n are s such that $\text{DEN}(s_n) = 0$ also called poles of the system; $\text{res}_n = \left. \frac{\text{NUM}(s)}{s^2 \text{DEN}'(s)} \right|_{s=s_n}$ are the residuals linked to the poles of the system; C_0 and C_1 are the residuals linked to the double pole in the origin of axis due to the linear ramp input;

$$\begin{aligned} C_0 &= \left[\frac{\text{NUM}(s)}{\text{DEN}(s)} \right]_{s=0} C_1 \\ &= \left[\frac{\text{NUM}'(s)\text{DEN}(s) - \text{NUM}(s)\text{DEN}'(s)}{\text{DEN}(s)^2} \right]_{s=0}; \\ \text{DEN}'(s) &= \frac{d}{ds} \text{DEN}(s); \quad \text{NUM}'(s) = \frac{d}{ds} \text{NUM}(s); \end{aligned}$$

Note that $\text{num}(z) \neq \text{ZT}[\text{NUM}(s)]$, $\text{den}(z) \neq \text{ZT}[\text{DEN}(s)]$.

In the Z-transform domain, we have:

$$G(z) = \frac{O(z)}{I(z)} = \frac{\frac{C_0 \Delta}{(1-z^{-1})^2} + \frac{C_1}{1-z^{-1}} + \sum \frac{\text{res}_n}{1-e^{s_n \Delta} z^{-1}}}{\frac{C_0 \Delta}{z(1-z^{-1})}} = \frac{\text{num}(z)}{\text{den}(z)} \quad (5)$$

where Δ is the sampling period.

$$\text{den}(z) = \prod (1 - e^{-s_n \Delta} z^{-1}) \quad (6)$$

where the sum and the product are extended to all the poles of the system.

Assuming the one-dimensional heat transfer in a homogeneous and isotropic layer with constant thickness L , letting be $\theta(x, t)$ the temperature and $q(x, t)$ the heat flux at the time t along x direction, the system equations that represent the thermal balance of the layer can be represented in a compact matrix notation:

$$\begin{bmatrix} T_e \\ Q_e \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \times \begin{bmatrix} T_i \\ Q_i \end{bmatrix} \quad (7)$$

where T_e is the LT of the temperature t_e of the external side of the layer, Q_e the LT of the heat flux q_e on the external side of

the layer, T_i the LT of the temperature t_i of the inner side of the layer, Q_i the LT of the heat flux q_i on the inner side of the layer, L the thickness of the layer and s is the Laplace variable.

$$a = \cosh\left(L\sqrt{\frac{s}{\alpha}}\right); \quad b = \frac{\sinh\left(L\sqrt{\frac{s}{\alpha}}\right)}{\lambda\sqrt{\frac{s}{\alpha}}};$$

$$c = \lambda\sqrt{\frac{s}{\alpha}}\sinh\left(L\sqrt{\frac{s}{\alpha}}\right); \quad d = a \quad (8)$$

where λ is the thermal conductivity (W/(m K)); ρ density (kg/m³); C_p specific heat (kJ/(kg K)) and $\alpha = \frac{\lambda}{\rho C_p}$ is thermal diffusivity (m²/s).

The non steady-state heat problems in multilayered walls can be described by extending the previous notation:

$$\begin{vmatrix} T_e \\ Q_e \end{vmatrix} = \begin{vmatrix} A(s) & B(s) \\ C(s) & D(s) \end{vmatrix} \times \begin{vmatrix} T_i \\ Q_i \end{vmatrix} \quad (9)$$

in which T and Q are, in this case, LT of the temperatures t_i and t_e and of the heat fluxes q_i and q_e in correspondence of the inner and outside surfaces of the composite wall, while A , B , C and D are the coefficients of the wall transmission matrix reached through the product of transmission matrixes, of each n_w layers forming the wall:

$$\begin{vmatrix} A(s) & B(s) \\ C(s) & D(s) \end{vmatrix} = \prod_{m=1}^{n_w} \begin{vmatrix} a_m & b_m \\ c_m & d_m \end{vmatrix} \quad (10)$$

$a_m b_m c_m d_m$ are calculated using equation (8).

By using the above relation, we can solve equations describing the thermal balance of a multilayered wall in the ZT domain:

$$Q_i(z) = \frac{1}{B(z)} T_e(z) - \frac{A(z)}{B(z)} T_i(z) \quad (11)$$

The numerical nature of the method is coupled to the impossibility to obtain the analytical identification of the infinite poles of the thermal system. The poles of the system have to lay on the negative part of the real axis in the complex domain, so, it is useful to make the substitution $\sqrt{s} = j\delta$ that makes us able to search poles only in the real numbers domain. Limiting the search to the first n poles near to the origin and making some manipulations, it is possible to link this method to the well known polynomial expressions of the Transfer Function Method.

The description of thermal behaviour of a thermal zone encompassed by n_w walls implies an increased complexity. The thermal balance of a wall bordering with the external environment can be better understood by Fig. 1.

The equation representing the thermal balance of the generic wall bordering with the external environment is:

$$q_{iK} S_K = f_K S_K (T_K - T_a) + S_K \sum_{J=1, J \neq K}^{n_w} r_{K,J} (T_K - T_J) - E_K \quad (12)$$

where I_{DK} is the beam solar radiation incident on the window belonging to the K th wall (W/m²), I_{dK} the diffuse solar

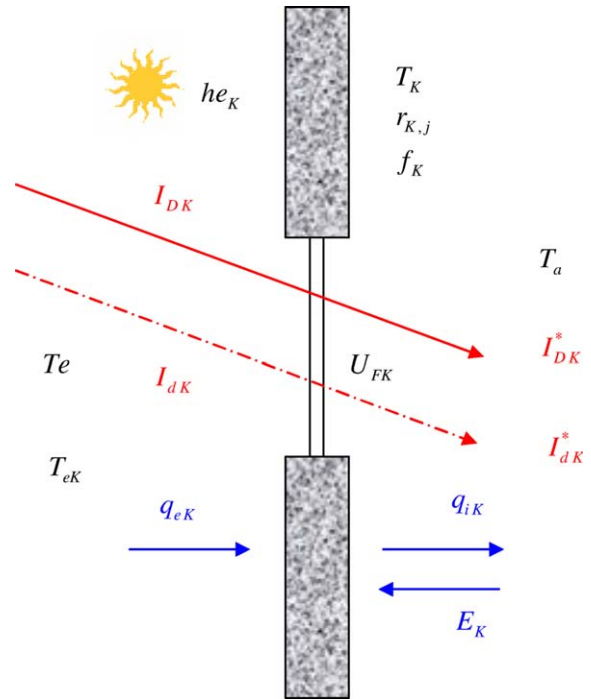


Fig. 1. Thermal balance of generic wall bordering with the external environment.

radiation incident on the window belonging to the K th wall (W/m²), I_{DK}^* the beam solar radiation emerging from the window belonging to the K th wall (W/m²), I_{dK}^* the diffuse solar radiation emerging from the window belonging to the K th wall (W/m²), q_{iK} the thermal flux emerging from the K th wall (W/m²), S_K the net surface of the K th wall (m²), f_K the internal convective factor of the K th wall (W/(m² K)), T_K the temperature of the inner surface of the K th wall (K), T_J the temperature of the inner surface of the J th wall (K), T_a the temperature of inner air (K), $r_{K,J}$ the radiative exchange coefficient between K th and of the J th wall (W/(m² K)) and E_K is the thermal flux incident on the K th wall coming from the internal environment (W).

At the same time, the thermal balance of wall K th can be written using the TFM formalism:

$$\begin{vmatrix} T_{eK} \\ q_{eK} \end{vmatrix} = \begin{vmatrix} 1 & \frac{1}{h_{eK}} \\ 0 & 1 \end{vmatrix} \times \begin{vmatrix} A_K & B_K \\ C_K & D_K \end{vmatrix} \times \begin{vmatrix} T_K \\ q_{iK} \end{vmatrix}$$

$$= \begin{vmatrix} A_K + \frac{C_K}{h_{eK}} & B_K + \frac{C_K}{h_{eK}} \\ C_K & D_K \end{vmatrix} \times \begin{vmatrix} T_K \\ q_{iK} \end{vmatrix} \quad (13)$$

where T_{eK} is the air-sol external temperature of the K th wall, taking into account the solar radiation (I_{DK} , I_{dK}) (K), q_{eK} the thermal flux entering in the K th wall (W/m²) and h_{eK} is the combined radiative-convective external coefficient on the external surface of the K th wall (W/(m² K)).

The last two equations can be merged into a unique expression:

$$\left[T_{eK} - \left(A_K + \frac{C_K}{h_{eK}} \right) T_K \right] S_K = \left(B_K + \frac{D_K}{h_{eK}} \right) \times \left[f_K S_K (T_K - T_a) + S_K \sum_{J=1, J \neq K}^{n_w} r_{K,J} (T_K - T_J) - E_K \right] \quad (14)$$

We can extend the above procedure for the generic internal walls that will be solicited by radiative exchange with the other surfaces, by the solar radiation coming through each window (I_{DK}^* and I_{dk}^*). The thermal balance of the inner air is represented by:

$$\sum_{K=1}^{n_w} f_K S_K (T_K - T_a) + \sum_{K=1}^{\text{all windows}} U_{FK} S_{FK} (T_e - T_a) + C_{Pa} \rho_a n V (T_e - T_a) = W_{CI} - W_{imp} + C_a \frac{dT_a}{d\tau} \quad (15)$$

where n is the hourly ventilation of the inner air, V the inner air volume (m^3), U_{FK} the exchange coefficient of the windows ($W/(m^2 K)$), S_{FK} the net surface of the K th windows (m^2), T_e the temperature of external air (K), C_{Pa} the specific heat of the air ($kJ/(kg K)$), ρ_a the density of the air (kg/m^3), C_a the thermal capacity of the air (kJ/K), W_{CI} the global thermal load (gain) (W) and W_{imp} is thermal gain due to HVAC (W).

Now, it is possible to write the global thermal balance of the whole room referred to three kinds of “knots”: generic walls bordering with the environment, with or without windows; generic internal walls; internal air.

For an example, room composed by six walls, walls 1, 2 and 3 bordering with the external environment

and with windows, walls 4, 5 and 6 generic internal walls, we can arrange the equations into a unique matrix form.

First row of the matrix:

$$\left[A_1 + \frac{C_1}{h_e} + \left(f_1 + \sum_{j=1, j \neq 1}^6 r_{1,j} \right) \left(B_1 + \frac{D_1}{h_e} \right) \right] S_1 T_1 - r_{1,2} \left(B_1 + \frac{D_1}{h_e} \right) S_1 T_2 - r_{1,3} \left(B_1 + \frac{D_1}{h_e} \right) S_1 T_3 - r_{1,4} \left(B_1 + \frac{D_1}{h_e} \right) S_1 T_4 - r_{1,5} \left(B_1 + \frac{D_1}{h_e} \right) S_1 T_5 - r_{1,6} \left(B_1 + \frac{D_1}{h_e} \right) S_1 T_6 - f_1 \left(B_1 + \frac{D_1}{h_e} \right) S_1 T_a = T_{e1} S_1 + \left(B_1 + \frac{D_1}{h_e} \right) E_1 \quad (16)$$

Second row of the matrix:

$$- r_{2,1} \left(B_2 + \frac{D_2}{h_e} \right) S_2 T_1 + \left[A_2 + \frac{C_2}{h_e} + \left(f_2 + \sum_{j=1, j \neq 2}^6 r_{2,j} \right) \left(B_2 + \frac{D_2}{h_e} \right) \right] S_2 T_2 - r_{2,3} \left(B_2 + \frac{D_2}{h_e} \right) S_2 T_3 - r_{2,4} \left(B_2 + \frac{D_2}{h_e} \right) S_2 T_4 - r_{2,5} \left(B_2 + \frac{D_2}{h_e} \right) S_2 T_5 - r_{2,6} \left(B_2 + \frac{D_2}{h_e} \right) S_2 T_6 - f_2 \left(B_2 + \frac{D_2}{h_e} \right) S_2 T_a = T_{e2} S_2 + \left(B_2 + \frac{D_2}{h_e} \right) E_2 \quad (17)$$

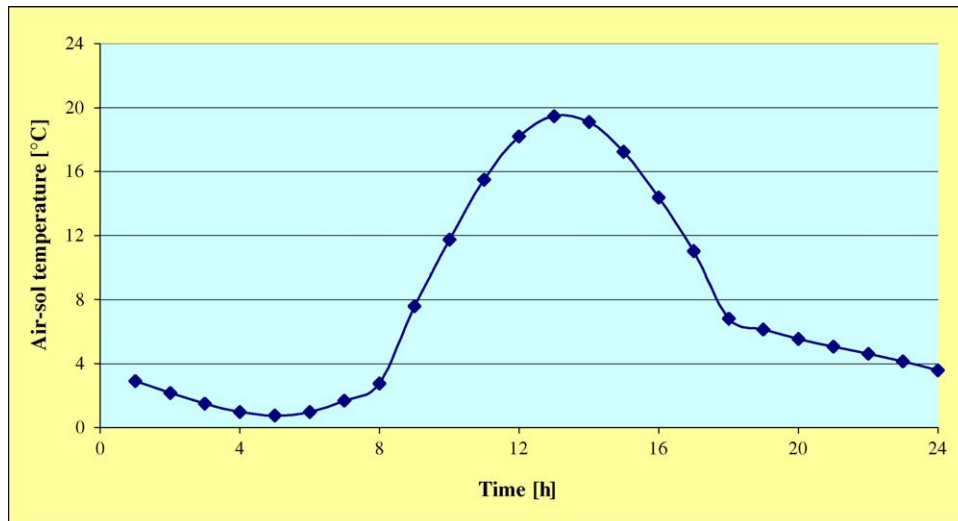


Fig. 2. Winter air-sol temperature.

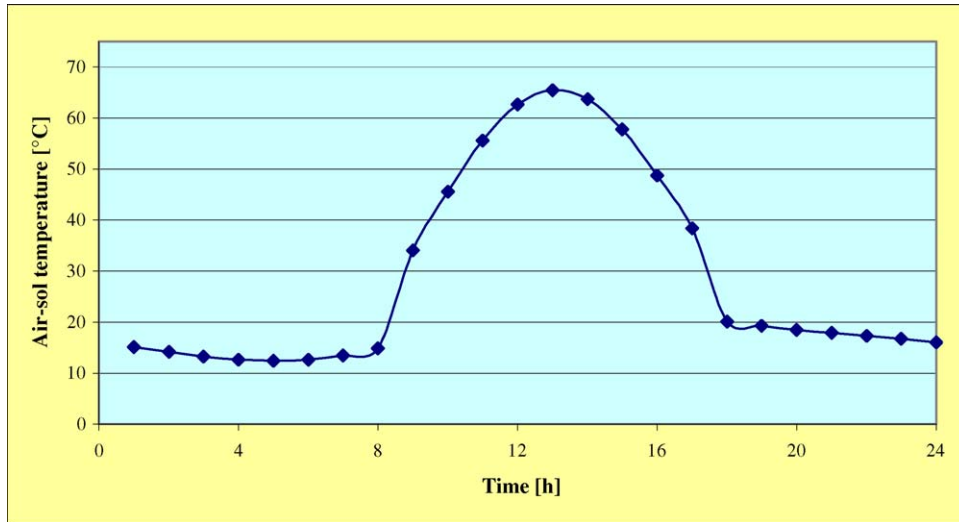


Fig. 3. Summer air-sol temperature.

Third row of the matrix:

$$\begin{aligned}
 & -r_{3,1} \left(B_3 + \frac{D_3}{h_e} \right) S_3 T_1 - r_{3,2} \left(B_3 + \frac{D_3}{h_e} \right) S_3 T_2 \\
 & + \left[A_3 + \frac{C_3}{h_e} + \left(f_3 + \sum_{j=1, j \neq 3}^6 r_{3,j} \right) \left(B_3 + \frac{D_3}{h_e} \right) \right] S_3 T_3 \\
 & - r_{3,4} \left(B_3 + \frac{D_3}{h_e} \right) S_3 T_4 - r_{3,5} \left(B_3 + \frac{D_3}{h_e} \right) S_3 T_5 \\
 & - r_{3,6} \left(B_3 + \frac{D_3}{h_e} \right) S_3 T_6 - f_3 \left(B_3 + \frac{D_3}{h_e} \right) S_3 T_a \\
 & = T_{e3} S_3 + \left(B_3 + \frac{D_3}{h_e} \right) E_3
 \end{aligned} \quad (18)$$

Fourth row of the matrix:

$$\begin{aligned}
 & -r_{4,1} D_4 S_4 T_1 - r_{4,2} D_4 S_4 T_2 - r_{4,3} D_4 S_4 T_3 \\
 & + \left[C_4 + D_4 \left(f_4 + \sum_{j=1, j \neq 4}^6 r_{4,j} \right) \right] S_4 T_4 \\
 & - r_{4,5} D_4 S_4 T_5 - r_{4,6} D_4 S_4 T_6 - f_4 D_4 S_4 T_a = D_4 E_4
 \end{aligned} \quad (19)$$

Fifth row of the matrix:

$$\begin{aligned}
 & -r_{5,1} D_5 S_5 T_1 - r_{5,2} D_5 S_5 T_2 - r_{5,3} D_5 S_5 T_3 - r_{5,4} D_5 S_5 T_4 \\
 & + \left[C_5 + D_5 \left(f_5 + \sum_{j=1, j \neq 5}^6 r_{5,j} \right) \right] S_5 T_5 - r_{5,6} D_5 S_5 T_6 \\
 & - f_5 D_5 S_5 T_a = D_5 E_5
 \end{aligned} \quad (20)$$

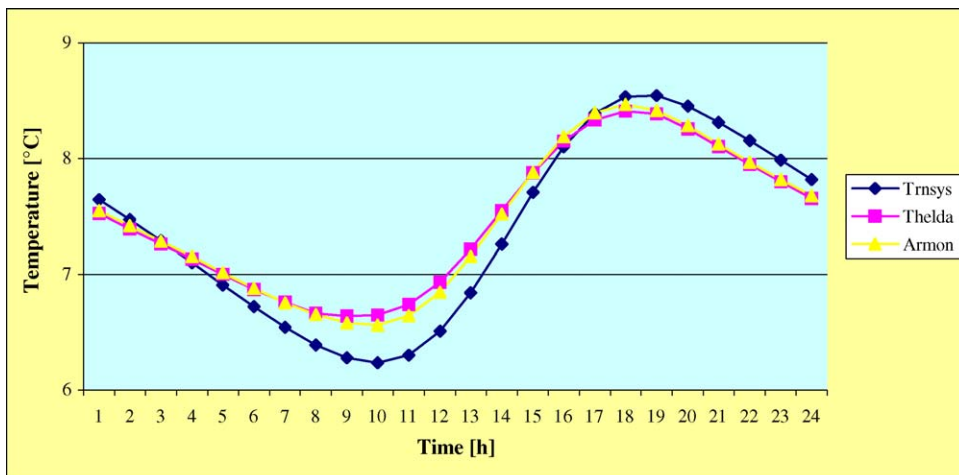


Fig. 4. Inner surface temperature of the wall 1; January.

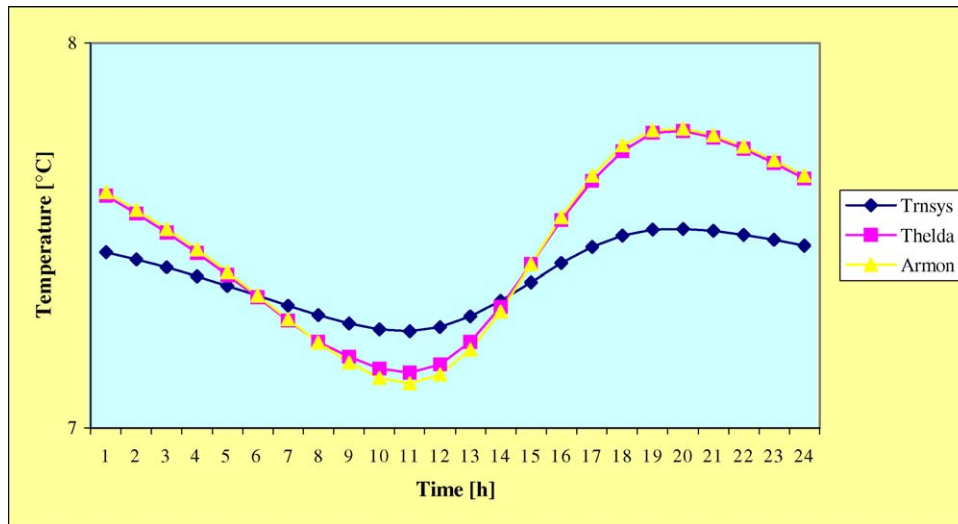


Fig. 5. Inner air temperature; January.

Sixth row of the matrix:

$$\begin{aligned}
 & -r_{6,1}D_6S_6T_1 - r_{6,2}D_6S_6T_2 - r_{6,3}D_6S_6T_3 - r_{6,4}D_6S_6T_4 \\
 & - r_{6,5}D_6S_6T_5 + \left[C_6 + D_6 \left(f_6 + \sum_{j=1, j \neq 6}^6 r_{6,j} \right) \right] S_6T_6 \\
 & - f_6D_6S_6T_a = D_6E_6
 \end{aligned} \quad (21)$$

Seventh row of the matrix:

$$\begin{aligned}
 & f_1S_1T_1 + f_2S_2T_2 + f_3S_3T_3 + f_4S_4T_4 + f_5S_5T_5 + f_6S_6T_6 \\
 & + \left[-sC_a - \sum_{j=1}^6 S_jF_j - \sum_{j=1}^3 S_{Fj}U_j - c_p\gamma nV \right] T_a \\
 & = - \left[\sum_{j=1}^3 S_{Fj}U_j - c_p\gamma nV \right] T_e - W_{CI} + W_{imp}
 \end{aligned} \quad (22)$$

Equations are in standard form because all of the unknowns are on one side, the constant terms are on the

other side and unknowns in each equation are aligned into the same column. By this way, the previous written system of linear equations can be synthetically expressed as:

$$(M) \times (V) = (U) \quad (23)$$

where the matrix of only the coefficients of the unknowns is (M) , (V) is the matrix of the unknown and (U) is the matrix of the known terms.

To find the transfer function $G(s)$, it is useful that the system is solicited by a unitary impulse input. In these conditions, the output of the system coincides with the transfer function.

Thanks to the linearity of the system, the generic output can be obtained by using superimposition of the effects as sum of the partial outputs induced by single input signals. By this way for each output can be defined as many partial transfer functions as the inputs of the system. These partial transfer functions can be identified

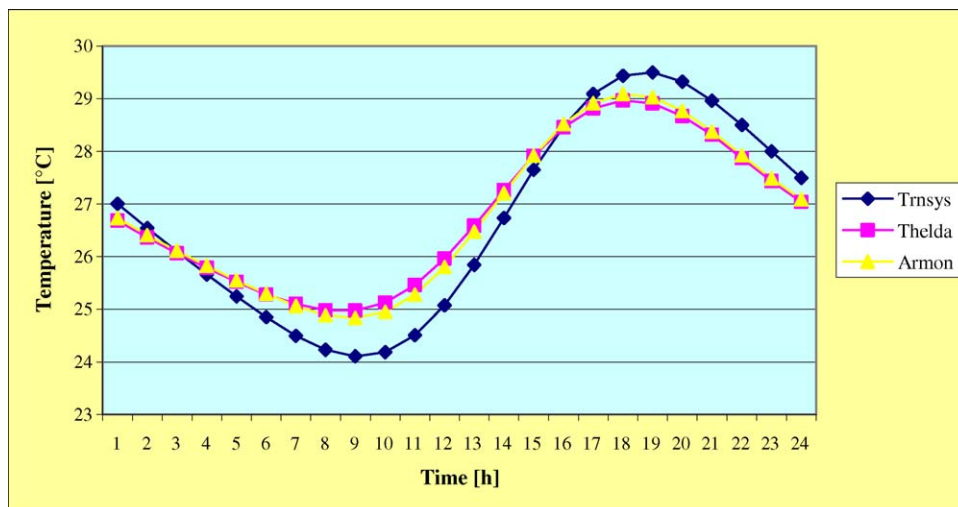


Fig. 6. Inner surface temperature; July.

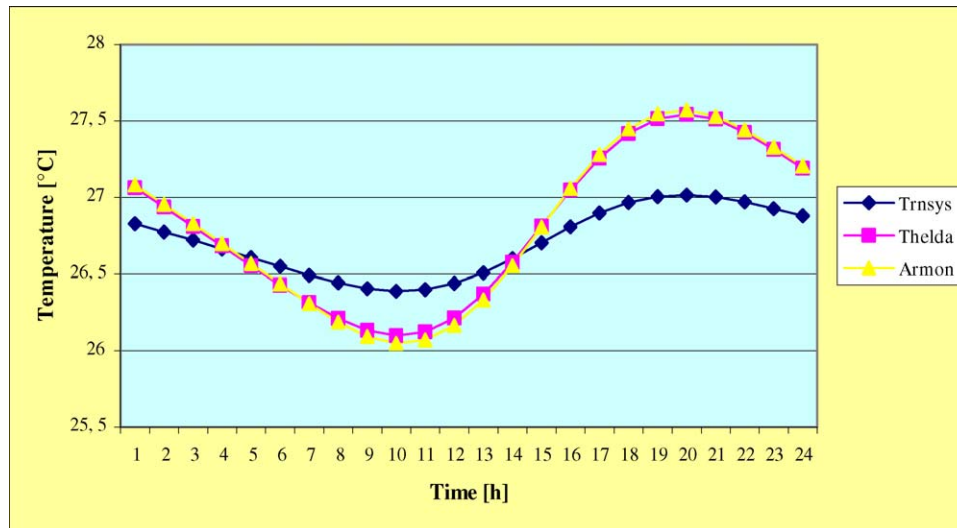


Fig. 7. Inner air temperature; July.

determining the generic output of the system imposing that:

- All of the terms composing the coefficients of the matrix (U) have to be zero except the term related to the input signal to which the partial transfer function is related to.
- Furthermore, the input signal ($T_{e,k}, I_{Dk}^*, \dots, W_{imp}$) to which the partial transfer function is referred, has to be unitary.

Referring to the above system's equations, the transfer function that links the generic temperature T_x to the generic input signal can be obtained solving the system's equation with the Cramer method:

$$G_{T_x, I_i(s)} = \frac{\det[M(T_x)]}{\det[M]} \quad (24)$$

where $\det[M(T_x)]$ is the determinant of the matrix (M) in which we have replaced the column related to the unknown T_x with the coefficients of the matrix (U) in which was deleted all the terms not referred to the x input. At the same time, all the terms of the matrix (U) linked to the x input are unitary. Assuming that even in this case, to obtain the ZT coefficients, we are employing linear ramp input, then the generic partial output caused by the unitary input will be:

$$\begin{aligned} T_{x,i}(s) &= \frac{1}{s^2} \frac{\det[M(T_x)]}{\det[M]} = \frac{1}{s^2} \frac{\text{NUM}(s)}{\text{DEN}(s)} \\ &= \frac{C_0}{s^2} + \frac{C_1}{s} + \sum_{n=1}^{\infty} \frac{\text{res}_n}{s - s_n} \end{aligned} \quad (25)$$

in which the poles s_n are obtained individuating the values of s that make null the determinant $\det[M]$. Once obtained the generic s_n , we can calculate with automatic procedure:

$$\text{res}_n = \left[\frac{\text{NUM}(s)}{s^2 \text{DEN}'(s)} \right]_{s=s_n} = \frac{\det[M(T_x)]|_{s=s_n}}{s_n^2 \frac{d}{ds} \{\det[M]\}|_{s=s_n}} \quad (26)$$

Analogously, it is possible to calculate C_0 and C_1 . Once obtained these coefficients, we can calculate the ZT of the generic partial output:

$$\begin{aligned} T_{x,i}(z) &= \frac{\text{num}(z)}{\text{den}(z)} I(z) \\ &= \frac{\text{num}_0 + \text{num}_1 z^{-1} + \text{num}_2 z^{-2} + \dots + \text{num}_n z^{-n}}{1 + \text{den}_1 z^{-1} + \text{den}_2 z^{-2} + \dots + \text{den}_n z^{-n}} \times I(z) \end{aligned} \quad (27)$$

in which:

$$\frac{\text{num}(z)}{\text{den}(z)} = \frac{\frac{C_0 \Delta}{(1-z^{-1})^2} + \frac{C_1}{1-z^{-1}} + \sum_{n=1}^{\infty} \frac{\text{res}_n}{1 - e^{s_n \Delta} z^{-1}}}{\frac{C_0 \Delta}{z(1-z^{-1})}} \quad (28)$$

Applying the definition of TFM and expanding the terms, we can write that:

$$\begin{aligned} &T_{x,i}(0) + T_{x,i}(\Delta)z^{-1} + T_{x,i}(2\Delta)z^{-2} \\ &+ \dots + T_{x,i}(n\Delta)z^{-n} + \dots \\ &= \frac{\text{num}_0 + \text{num}_1 z^{-1} + \text{num}_2 z^{-2} + \dots + \text{num}_n z^{-n}}{1 + \text{den}_1 z^{-1} + \text{den}_2 z^{-2} + \dots + \text{den}_n z^{-n}} \cdot [I_i(0) \\ &+ I_i(\Delta)z^{-1} + I_i(2\Delta)z^{-2} + \dots + I_i(n\Delta)z^{-n} + \dots] \end{aligned}$$

Table 1
Description of the layer composing the wall 1 facing south

Layer	Description	Thickness (m)	Density (kg/m ³)	Specific heat (kJ/(kg K))	Conductivity (W/(m K))	Thermal resistance (m ² K/W)
1	Plaster	0.02	1602	0.84	0.727	0.026
2	Sandstone	0.2	2300	0.7	1.7	0.117
3	Plaster	0.02	1602	0.84	0.727	0.026

Table 2
Description of the layer composing the internal wall 2

Layer	Description	Thickness (m)	Density (kg/m ³)	Specific heat (kJ/(kg K))	Conductivity (W/(m K))	Thermal resistance (m ² K/W)
1	Plaster	0.02	1602	0.84	0.727	0.026
2	Sandstone	0.4	2300	0.7	1.7	0.235
3	Plaster	0.02	1602	0.84	0.727	0.026

After a long manipulation of the above equation, we obtain the recursive formula:

$$T_{x,i}(n\Delta) = \sum_{j=1}^n \text{num}_j I_i[(n-j)\Delta] - \sum_{j=1}^n \text{den}_j T_{x,i}[(n-j)\Delta] \quad (29)$$

valid for the generic time $n\Delta$.

The time profile of the generic output T_x can be calculated superimposing all the partial outputs:

$$T_x(n\Delta) = \sum_{i=1}^{\text{all inputs}} T_{x,i}(n\Delta) \quad (30)$$

Transfer function coefficients have to be calculated for each different pair of input–output, e.g., air–sol temperature and the inner air temperature, inner thermal loads and the temperature of inner surface 5 and so on. All of the outputs referred to the same physical knot are later summed to obtain the global response.

4. Evaluation of the reliability of the proposed method by comparison with harmonic solution

In order to evaluate the reference response if we want to control the reliability of the coefficients sets evaluated by means of diverse input signals, we have to:

- Use input signals with the typical time variation of the physical phenomena and then avoiding steps, ramps, etc.
- Adopt an evaluating approach which differs from that employed to calculate ZT coefficients sets, and then exclude Laplace Transform.

We carried out a comparison between simulation data obtained from Fourier steady-state algorithm and those obtained from TFM [12]. To perform this comparison, it has been necessary, to define for any generic input signal, a rule of interpolation able to reproduce a physic realistic behaviour starting from the sampled data.

So, the input signal has been represented by the function $X(t)$, which is not discrete within the generic interval of sample t_n, t_{n+1} defined by the expression:

$$X(t) = at^3 + bt^2 + ct + d \quad (31)$$

in which the coefficients a, b, c, d are calculated imposing the following constraints:

- (1) the $X(t)$ assumes the values $X(t_n)$ and $X(t_{n+1})$;
- (2) the tangent to the curve at the point $[t_n, X(t_n)]$, forms equal angles with the segments joining $X(t)$ to $X(t_{n-1})$ and $X(t_n)$ to $X(t_{n+1})$.

In periodic steady-state, any wave $f(t)$ with period T can be analysed into the sum of sinusoidal waves of different frequencies:

$$f(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} [a_n \cos(n\omega t) + b_n \sin(n\omega t)] \quad (32)$$

Supposing that $f(t)$ is known in correspondence to the N instants in which the period T was subdivided, coefficients are:

$$\begin{cases} a_0 = \frac{2}{N} \sum_{k=0}^{N-1} f\left(\frac{kt}{N}\right) \\ a_n = \frac{2}{N} \sum_{k=0}^{N-1} f\left(\frac{kt}{N}\right) \cos\left(\frac{2\pi kn}{N}\right) \\ b_n = \frac{2}{N} \sum_{k=0}^{N-1} f\left(\frac{kt}{N}\right) \sin\left(\frac{2\pi kn}{N}\right) \end{cases} \quad (33)$$

Then, calculating the time response to each harmonic component and recombining all of them, one can obtain the system response to the generic $f(t)$.

The calculated temperatures have been compared to those obtained by Fourier's analysis in periodic steady-state conditions. To exit from transient response, the calculation with ZT is repeated until the signal is stabilised [13].

Table 3
Description of the layer composing the internal walls 3 and 4

Layer	Description	Thickness (m)	Density (kg/m ³)	Specific heat (kJ/(kg K))	Conductivity (W/(m K))	Thermal resistance (m ² K/W)
1	Plaster	0.02	1602	0.84	0.727	0.026
2	Sandstone	0.42	2300	0.7	1.7	0.246
3	Plaster	0.02	1602	0.84	0.727	0.026

Table 4

Description of the layer composing the floor and the roof

Layer	Description	Thickness (m)	Density (kg/m ³)	Specific heat (kJ/(kg K))	Conductivity (W/(m K))	Thermal resistance (m ² K/W)
1	Concrete	0.05	2200	0.92	1.48	0.034
2	Concrete	0.3	1500	0.88	0.66	0.454
3	Concrete	0.05	2200	0.92	1.48	0.034

5. Case study

We simulated the thermal behaviour of three different rooms to test the reliability of the method. We employed as input signals some weather variables such as air–sol temperature (air–sol temperature is a fictive temperature employed to link conduction and radiative effects) and solar radiation (beam and diffuse).

We reported the results of the performed simulations for two different input signals:

- Winter air–sol temperature referred to January in the city of Palermo, Fig. 2.
- Summer air–sol temperature referred to July in the city of Palermo, Fig. 3.

The definition of air–sol temperature is:

$$t_{\text{air-sol}} = t_a + \frac{\alpha I + \varepsilon(R_s - R_L)}{h} \quad (34)$$

where t_a is the air temperature (°C), α the absorptivity of the surface, I the solar radiation over the surface (W/m²), h the heat exchange coefficient (convective and radiative) (W/(m² °C)), ε the emissivity of the surface, R_s the low-temperature radiative energy over the surface, R_L the low-temperature radiative energy emitted by a blackbody at the same temperature of the surface,

$$R_s = [1.06 \times \sigma(t_a + 273.15)^4 - 119] \times (1 - 0.84c) + 0.84c \quad (35)$$

$$R_L = \sigma \times (t_s + 273.15)^4 \quad (36)$$

where σ is the Stephan–Boltzmann constant, t_s the temperature of the surface and c the fraction of sky obscured by clouds.

By this definition, air–sol temperature even describes the radiative exchange of the surface with the sky during the night.

6. Description of the simulations performed on room

The room A used for the analysis is composed of six walls, one bordering the external environment (21.673 m²) facing south, three internal walls (34.69 m², 21.673 m² and 34.69 m²) roof and floor (72.065 m²). The volume is 129 m³ and every wall is composed of a sandstone layer in the middle and inside-outside plaster (0.02 m). Floor and roof are constituted by concrete and to simplify the

simulation, we do not insert windows or inner heat sources. In the following figures, we will refer to our method with the name thermal elements dynamic analysis (THELDA) that is the name of the software developed to test the method [8–10]. Results obtained by harmonic solution are grouped under the name ARMON [11]. Furthermore, we simulated the room with TRNSYS, a commercial tool for thermal simulation. Results of simulations are summarised in Figs. 4–7 (Tables 1–4).

7. Conclusions

We have presented an algorithm that uses the Z-transform mathematical operator to solve in a strict mathematical form a system of linear differential equations representing the thermal dynamic behaviour of a single thermal zone, composed by multilayered walls.

To prevent and solve the mathematical problems rising from “ASHRAE” method, we developed a procedure to determine the TFs of a single thermal zone by using Z-transform.

The method is very flexible and could be adopted to calculate the transfer function coefficients able to simulate the thermal behaviour of a room in free floating. Knowing the transfer function coefficients, it is possible to simulate the dynamic profile of each inner surfaces temperature and furthermore of the inner air temperature.

The proposed algorithm is fully described granting maximum clarity. The explicitness of all steps of the calculus make possible the definition of a method that is able to vary all of the calculus parameters such as sampling g period, number of roots, number of poles or number of coefficients. By this way, it is possible to carry out a sensitivity analysis to better understand the relative influence of these parameters in generating the model outputs. This flexibility of the method is a fundamental characteristic to improve the performance of thermal building simulation software.

The ability to solve the system equations describing the thermal balance of a thermal zone without any simplification to reduce the number of variables and equations is an innovative feature of this algorithm. For example, one of the most diffused software for thermal building simulation, TRNSYS, uses the “network temperature” to simplify the thermal balance and consequently to simplify the numerical resolution employing a single equation [12].

The new procedure allows to achieve a drastic improvement in the reliability of the simulations results [13–15]. Our

procedure is able to find all the coefficients that we require, independently from their order or from thermal inertia of the room or from thermal inertia of the walls delimiting the room. Knowing the transfer function coefficients, it is possible to simulate the dynamic profile of each inner surfaces temperature and furthermore of the inner air temperature.

To assess the reliability of the algorithm, we carried out a comparison between simulation data obtained from our method, from Fourier steady-state algorithm and those obtained from TRNSYS.

References

- [1] M. Gough, A Review of New Techniques in Building Energy and Environmental Modelling, Report for Contract BREA-42, Building Research Establishment, 1999.
- [2] V.I. Hanby, A.J. Dil, Stochastic modelling of building heating & cooling systems, BSER&T 16 (4) (1995).
- [3] H.S. Carslaw, J.C. Jager, Conduction of Heat in Solid, Oxford University Press, 1959.
- [4] E. Palomo, G. Lefebvre, Stochastic simulations against deterministic ones: advantages and drawbacks, in: Proceedings of Building Simulation, 1995.
- [5] D. Haltrecht, R. Zmeureanu, I. Beausoleil-Morrison, Defining the Methodology for the Next-Generation HOT2000TM Simulator, in: Proceedings of Building Simulation 99, Kyoto, Japan, September 13–15, 1999, vol. 1, pp. 61–68.
- [6] A.D. McKinley, G.P. Mitalas, Room thermal transfer functions—computer programs to calculate Z-Transfer function coefficients for rooms, Computer Program No. 52, Version A01, Division of Building Research, Ottawa, Canada, 1983.
- [7] G.P. Mitalas, Room dynamic thermal response, in: Proceedings of The Use of Computers for Environmental Engineering Related to Buildings Conference, Tokyo, Japan, 1983.
- [8] C. Giaconia, A. Orioli, On the reliability of conduction transfer function coefficients of walls, in: Applied Thermal Engineering 20, Pergamon Press, Cambridge, UK, 2000.
- [9] G. Beccali, M. Cellura, V. Lo Brano, A database of Z-transform coefficient for Mediterranean building typologies, in: Proceedings of BS2001, Rio de Janeiro, Brazil, 2001.
- [10] M. Cellura, V. Lo Brano, A. Orioli, Thermal dynamic analysis of buildings: software THELDA, in: Proceedings of Clima 2000, Naples, Italy, 2000.
- [11] Lara Pasqualetto, Ramu Zmeureanu, Paul Fazio, A case study of validation of an energy analysis program: Micro-Doe2.1 E; Building and Environment, Pergamon Press, 1998.
- [12] S.A. Klein, W.A. Beckmann, J.A. Duffie, TRNSYS—a transient simulation program, ASHRAE Trans. 82 (1976) 623.
- [13] G. Beccali, M. Cellura, V. Lo Brano, A. Orioli, An improved algorithm for thermal dynamic simulation of walls using Z-transform coefficients, in: Proceedings of Eleventh International Conference on Computational Methods and Experimental Measurements, Halkidiki, Greece, 2003.
- [14] M. Cellura, L. Giarrè, V. Lo Brano, A. Orioli, Thermal dynamic models using Z-transform coefficients: an algorithm to improve the reliability of simulations, in: Proceedings of Building Simulation 2003, IBPSA International Conference and Exhibition on Building Simulation, Eindhoven, the Netherlands, 2003.
- [15] G. Beccali, M. Cellura, V. Lo Brano, A. Orioli, Is the transfer function method reliable in a European building context? A theoretical analysis and a case study in the south of Italy, Applied Thermal Engineering 25 (2–3) (February 2005) 341–357.