

qreals: computing q-deformed real numbers

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What the engine computes

An ordinary integer n has a standard q -analogue

$$[n]_q = 1 + q + q^2 + \cdots + q^{n-1},$$

which returns to n when $q = 1$. Morier-Genoud and Ovsienko (MGO) extended this from integers to rationals, and then to every real number, using continued fractions. For a rational p/s the result $[p/s]_q$ is an exact rational function of q ; for an irrational x it is a power series in q with integer coefficients. `qreals` provides both:

- `q_rational(p, s)` returns the exact $[p/s]_q$. For instance $[3/2]_q = (q^2 + q + 1)/(q + 1)$.
- `q_real_truncated(x, N)` returns the first N integer coefficients of $[x]_q$. The series for $[\pi]_q$, for example, begins $1, 1, 1, 0, 0, \dots$

How it works

Three steps turn a real number into its q -series.

1. **Continued fraction.** Expand $x = [a_1, a_2, a_3, \dots]$. The depth is not a free parameter. MGO Proposition 1.1 says that stopping once the partial quotients sum to at least $N + 1$ fixes exactly that many stable coefficients, so the engine reads quotients until the running sum clears N .
2. **Even-length form.** The MGO formula needs an even-length regular continued fraction, so an odd tail is rewritten by one of two rules: split a final entry, or absorb a trailing one.
3. **The MGO formula.** The even-length continued fraction is folded from the inside out, alternating $[a_i]_q$ and $[a_i]_{q^{-1}}$ terms. The folding runs over a truncated Laurent series in q with exact integer coefficients; the one nontrivial operation, series inversion, uses Newton's iteration over the integers, so nothing is lost to rounding.

For a rational the continued fraction terminates, and the truncated series then matches the Taylor expansion of the exact rational function coefficient for coefficient, which the test suite verifies.

Worked example

Computing $[\pi]_q$ to twelve coefficients with the engine gives

$$1, 1, 1, 0, 0, 0, 0, 0, 0, 1, 0, \dots$$

These are exact integers: the series path holds each coefficient as a Python int and inverts series with Newton's iteration over the integers, so nothing is lost to rounding. Asking for more coefficients never changes the ones already returned, which is the stability guarantee of MGO Proposition 1.1.

For a rational such as $[3/2]_q$ the continued fraction terminates, and the truncated series matches the Taylor expansion of the exact rational function coefficient for coefficient. The test suite checks both facts.

See `README.md` for the full API, and Morier-Genoud and Ovsienko, “q-deformed rationals and q-continued fractions”, Forum Math. Sigma 8 (2020), e13, for the construction.