

Certificate for $[\text{sqrt}(2)]_{-q}$ (first 13 coefficients)

qreals certificate

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Input: the real number $x = \text{sqrt}(2)$. The continued fraction is truncated to the depth that locks the first 13 coefficients (MGO Proposition 1.1).

(a) Continued fraction and even-length MGO form

Regular continued fraction: $[1, 2, 2, 2, 2, 2, 2, 2]$.

Even-length MGO form: $[1, 2, 2, 2, 2, 2, 2, 2]$.

The even-length form evaluates to the convergent $\frac{577}{408}$.

(b) MGO formula folded step by step

Odd positions carry $[a]_q$ with q^a above; even positions carry $[a]_{q^{-1}}$ with q^{-a} above (MGO eqn. 1.1).
innermost term a_8 = 2:

$$\frac{q+1}{q}$$

fold in a_7 = 2:

$$\frac{q^3 + q^2 + 2q + 1}{q + 1}$$

fold in a_6 = 2:

$$\frac{q^5 + 2q^4 + 3q^3 + 3q^2 + 2q + 1}{q^5 + q^4 + 2q^3 + q^2}$$

fold in a_5 = 2:

$$\frac{q^7 + 2q^6 + 5q^5 + 6q^4 + 6q^3 + 5q^2 + 3q + 1}{q^5 + 2q^4 + 3q^3 + 3q^2 + 2q + 1}$$

fold in a_4 = 2:

$$\frac{q^9 + 3q^8 + 7q^7 + 11q^6 + 13q^5 + 13q^4 + 11q^3 + 7q^2 + 3q + 1}{q^9 + 2q^8 + 5q^7 + 6q^6 + 6q^5 + 5q^4 + 3q^3 + q^2}$$

fold in a_3 = 2:

$$\frac{q^{11} + 3q^{10} + 9q^9 + 16q^8 + 24q^7 + 29q^6 + 29q^5 + 25q^4 + 18q^3 + 10q^2 + 4q + 1}{q^9 + 3q^8 + 7q^7 + 11q^6 + 13q^5 + 13q^4 + 11q^3 + 7q^2 + 3q + 1}$$

fold in a_2 = 2: The value is the ratio $N(q)/D(q)$ with:

$$N(q) = q^{13} + 4q^{12} + 12q^{11} + 25q^{10} + 41q^9 + 56q^8 + 65q^7 + 65q^6 + 56q^5 + 41q^4 + 25q^3 + 12q^2 + 4q + 1$$

$$D(q) = q^{13} + 3q^{12} + 9q^{11} + 16q^{10} + 24q^9 + 29q^8 + 29q^7 + 25q^6 + 18q^5 + 10q^4 + 4q^3 + q^2$$

fold in a_1 = 1: The value is the ratio $N(q)/D(q)$ with:

$$N(q) = q^{14} + 4q^{13} + 13q^{12} + 28q^{11} + 49q^{10} + 70q^9 + 85q^8 + 90q^7 + 83q^6 + 66q^5 + 45q^4 + 26q^3 + 12q^2 + 4q + 1$$

$$D(q) = q^{13} + 4q^{12} + 12q^{11} + 25q^{10} + 41q^9 + 56q^8 + 65q^7 + 65q^6 + 56q^5 + 41q^4 + 25q^3 + 12q^2 + 4q + 1$$

Result. $[577/408]_q$ (the convergent, whose Taylor expansion gives $[x]_q$): The value is the ratio $N(q)/D(q)$ with:

$$N(q) = q^{14} + 4q^{13} + 13q^{12} + 28q^{11} + 49q^{10} + 70q^9 + 85q^8 + 90q^7 + 83q^6 + 66q^5 + 45q^4 + 26q^3 + 12q^2 + 4q + 1$$

$$D(q) = q^{13} + 4q^{12} + 12q^{11} + 25q^{10} + 41q^9 + 56q^8 + 65q^7 + 65q^6 + 56q^5 + 41q^4 + 25q^3 + 12q^2 + 4q + 1$$

Taylor coefficients:

$$1 + q^3 - 2q^5 + q^6 + 4q^7 - 5q^8 - 7q^9 + 18q^{10} + 7q^{11} - 55q^{12} + O(q^{13})$$

(c) Independent cross-checks

- **[pass]** the first 13 coefficients are unchanged when 17 are asked for
- **[pass]** $[x+1]_q$ computed from its own continued fraction equals $q^*[x]_q + 1$
- **[na]** the series $[\text{sqrt}(2)]_q$ diverges at $q=1$; $q=1$ is for rationals
- **[na]** $\text{sqrt}(2)$ is irrational, so the continued fraction does not terminate

Summary: verified: truncation stable to 13, shift law $[x+1]_q = q[x]_q + 1$; n/a: $q=1$ specialisation; exact rational function.

References

S. Morier-Genoud and V. Ovsienko, q -deformed rationals and q -continued fractions, Forum Math. Sigma **8** (2020), e13. Per-function theorem and check mapping: `docs/CORRECTNESS.md`.

This is a human-auditable derivation, not a formal machine proof; every line above is checkable by hand.