

Certificate for $[\text{sqrt}(2)]_q$ (first 12 coefficients)

qreals certificate

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Input: the real number $x = \text{sqrt}(2)$. The continued fraction is truncated to the depth that locks the first 12 coefficients (MGO Proposition 1.1).

(a) Continued fraction and even-length MGO form

Regular continued fraction: $[1, 2, 2, 2, 2, 2, 2]$.

Even-length MGO form: $[1, 2, 2, 2, 2, 2, 1, 1]$.

The even-length form evaluates to the convergent $\frac{239}{169}$.

(b) MGO formula folded step by step

Odd positions carry $[a]_q$ with q^a above; even positions carry $[a]_{q^{-1}}$ with q^{-a} above (MGO eqn. 1.1).
innermost term **a_8 = 1**:

$$1$$

fold in a_7 = 1:

$$q + 1$$

fold in a_6 = 2:

$$\frac{q^3 + 2q^2 + q + 1}{q^3 + q^2}$$

fold in a_5 = 2:

$$\frac{q^5 + 2q^4 + 3q^3 + 3q^2 + 2q + 1}{q^3 + 2q^2 + q + 1}$$

fold in a_4 = 2:

$$\frac{q^7 + 3q^6 + 5q^5 + 6q^4 + 6q^3 + 5q^2 + 2q + 1}{q^7 + 2q^6 + 3q^5 + 3q^4 + 2q^3 + q^2}$$

fold in a_3 = 2:

$$\frac{q^9 + 3q^8 + 7q^7 + 11q^6 + 13q^5 + 13q^4 + 11q^3 + 7q^2 + 3q + 1}{q^7 + 3q^6 + 5q^5 + 6q^4 + 6q^3 + 5q^2 + 2q + 1}$$

fold in a_2 = 2:

$$\frac{q^{11} + 4q^{10} + 10q^9 + 18q^8 + 25q^7 + 29q^6 + 29q^5 + 24q^4 + 16q^3 + 9q^2 + 3q + 1}{q^{11} + 3q^{10} + 7q^9 + 11q^8 + 13q^7 + 13q^6 + 11q^5 + 7q^4 + 3q^3 + q^2}$$

fold in a_1 = 1: The value is the ratio $N(q)/D(q)$ with:

$$N(q) = q^{12} + 4q^{11} + 11q^{10} + 21q^9 + 31q^8 + 38q^7 + 40q^6 + 36q^5 + 27q^4 + 17q^3 + 9q^2 + 3q + 1$$

$$D(q) = q^{11} + 4q^{10} + 10q^9 + 18q^8 + 25q^7 + 29q^6 + 29q^5 + 24q^4 + 16q^3 + 9q^2 + 3q + 1$$

Result. $[239/169]_q$ (the convergent, whose Taylor expansion gives $[x]_q$): The value is the ratio $N(q)/D(q)$ with:

$$N(q) = q^{12} + 4q^{11} + 11q^{10} + 21q^9 + 31q^8 + 38q^7 + 40q^6 + 36q^5 + 27q^4 + 17q^3 + 9q^2 + 3q + 1$$

$$D(q) = q^{11} + 4q^{10} + 10q^9 + 18q^8 + 25q^7 + 29q^6 + 29q^5 + 24q^4 + 16q^3 + 9q^2 + 3q + 1$$

Taylor coefficients:

$$1 + q^3 - 2q^5 + q^6 + 4q^7 - 5q^8 - 7q^9 + 18q^{10} + 7q^{11} + O(q^{12})$$

(c) Independent cross-checks

- **[pass]** the first 12 coefficients are unchanged when 16 are asked for
- **[pass]** $[x+1]_q$ computed from its own continued fraction equals $q^*[x]_q + 1$
- **[na]** the series $[\text{sqrt}(2)]_q$ diverges at $q=1$; $q=1$ is for rationals
- **[na]** $\text{sqrt}(2)$ is irrational, so the continued fraction does not terminate

Summary: verified: truncation stable to 12, shift law $[x+1]=q[x]+1$; n/a: $q=1$ specialisation; exact rational function.

References

S. Morier-Genoud and V. Ovsienko, q-deformed rationals and q-continued fractions, Forum Math. Sigma **8** (2020), e13. Per-function theorem and check mapping: `docs/CORRECTNESS.md`.

This is a human-auditable derivation, not a formal machine proof; every line above is checkable by hand.